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On Orthogonal Polynomials*

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Let $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\gamma_n > 0\}_{n=0}^{\infty}$ be given sequences of real members. Putting $p_{-1} = 0$, $p_0 = \gamma_0$ and defining p_n for $n = 1, 2,...$ by

$$xp_{n-1}(x) = \frac{\gamma_{n-1}}{\gamma_n} p_n(x) + \alpha_{n-1} p_{n-1}(x) + \frac{\gamma_{n-2}}{\gamma_{n-1}} p_{n-2}(x)$$

we obtain a system of polynomials $\{p_n\}_{n=0}^{\infty}$ which by a result of J. Favard (see e.g. [2]) is orthonormal with respect to some positive measure $d\alpha$ acting on the real line. Let

$$c_n = \left| 1 - 2\frac{\gamma_{n-1}}{\gamma_n} \right| + 2\,|\alpha_{n-1}| + \left| 1 - 2\frac{\gamma_{n-2}}{\gamma_{n-1}} \right|.$$

It has been shown in [3] that under the assumption

$$\sum_{n=0}^{\infty} c_n < \infty \tag{1}$$

the measure $d\alpha$ can be written as

$$d\alpha(x) = \alpha'(x)\,dx + \sum \{\text{jumps outside } (-1, 1)\},$$

where α' is positive and continuous on $(-1, 1)$ and α' vanishes outside $[-1, 1]$. At the present time it is not clear, assuming (1), how α' behaves near -1 and 1. In case of the Chebyshev polynomials ($\alpha_n = 0$ for $n = 0, 1,..., \gamma_0 = \gamma_1 = 1$ and $\gamma_n = 2^{n-1}$ for $n = 2, 3,...$) α' is not continuous at -1 and 1. For the Chebychev polynomials of the second kind ($\alpha_n = 0$ and $\gamma_n = 2^n$ for $n = 0, 1,...$) α' is not positive at -1 and 1. From the work of G. Szegö (see, e.g., [4]) follows clearly that the measures $d\alpha$ for which

$$\int_{-\pi}^{\pi} \log \alpha'(\cos\theta)\,d\theta > -\infty \tag{2}$$

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play a very important role in the theory of orthogonal polynomials. Therefore it is natural to ask if (1) implies (2). It is easy to see that under the assumption $\text{supp}(d\alpha) = [-1, 1]$ this is indeed the case ([3]). Otherwise the question is still open. It was proved in [3] that

$$\sum_{n=0}^{\infty} nc_n < \infty \tag{3}$$

implies

$$\alpha'(x) \geqslant \text{const}(1 - x^2)^{1/2}$$

for $-1 \leqslant x \leqslant 1$. Hence (2) follows from (3). K. M. Case conjectured in [1] that (2) holds whenever

$$\limsup_{n\to\infty} n^2 c_n < \infty.$$

The purpose of this note is to show that the weaker condition

$$\sum_{n=0}^{\infty} (n+1)c_n \leqslant A \log(m+1) \qquad (m = 1, 2, \ldots) \tag{4}$$

not only implies (2) but also gives a pointwise estimate for α'. We will see that, assuming (4), $\log \alpha'$ is quite well-behaved. Our plan is the following. First, using an absolutely elementary method, we obtain an estimate for $|p_n|$. This method is somewhat miraculous since we establish an inequality which improves itself when applied repeatedly. Having a bound for $|p_n|$, the corresponding estimate for α' follows from a result in [3].

THEOREM. *Suppose that* (4) *holds with a suitable constant* $A > 0$. *Then there exist positive constants* A_1, A_2 *depending only on* A *and* $\inf_n \gamma_{n-1}/\gamma_n$ *such that*

$$|p_n(x)| \leqslant A_1(1 - x^2)^{-A_2} \qquad (-1 \leqslant x \leqslant 1) \tag{5}$$

for $n = 1, 2, \ldots$ *and*

$$\alpha'(x) \geqslant A_1^{-1}(1 - x^2)^{A_2} \qquad (-1 \leqslant x \leqslant 1). \tag{6}$$

Proof. Let $x \in [-1, 1]$ and put $x = \cos\theta$ where $0 \leqslant \theta \leqslant \pi$. Define ϕ_n by

$$\phi_n(\theta) = p_n(x) - e^{i\theta} p_{n-1}(x).$$

Then

$$\phi_n(\theta) - e^{-i\theta}\phi_{n-1}(\theta) = p_n(x) - 2xp_{n-1}(x) + p_{n-2}(x)$$

and, by the recurrence formula,

$$\phi_n(\theta) - e^{-i\theta}\phi_{n-1}(\theta)$$
$$= \left[1 - 2\frac{\gamma_{n-1}}{\gamma_n}\right] p_n(x) - 2\alpha_{n-1} p_{n-1}(x) + \left[1 - 2\frac{\gamma_{n-2}}{\gamma_{n-1}}\right] p_{n-2}(x). \quad (7)$$

Consequently

$$|\phi_n(\theta) - e^{-i\theta}\phi_{n-1}(\theta)| \leqslant c_n \sum_{k=n-2}^{n} |p_k(x)|.$$

Using again the recurrence formula, we obtain

$$\sum_{k=n-2}^{n} |p_k(x)| \leqslant K \sum_{k=M-1}^{M} |p_k(x)| \qquad (M = n-1, n) \quad (8)$$

where K depends only on $\sup_n \alpha_n$, $\inf_n \gamma_{n-1}/\gamma_n$ and $\sup_n \gamma_{n-1}/\gamma_n$. Furthermore, from the definition of ϕ_n follows that

$$(1 - x^2)^{1/2} |p_n(x)| \leqslant |\phi_n(\theta)|, \qquad (1 - x^2)^{1/2} |p_{n-1}(x)| \leqslant |\phi_n(\theta)|. \quad (9)$$

Therefore

$$|\phi_n(\theta) - e^{-i\theta}\phi_{n-1}(\theta)| \leqslant 2Kc_n(1 - x^2)^{-1/2} \max_{|x|\leqslant 1} |\phi_{n-1}(\theta)|.$$

Recall that $\phi_n - e^{i\theta}\phi_{n-1}$ is a polynomial of degree n in x. Thus by a theorem of S. Bernstein,

$$\max_{|x|\leqslant 1} |\phi_n(\theta) - e^{-i\theta}\phi_{n-1}(\theta)| \leqslant 2Kc_n(n+1) \max_{|x|\leqslant 1} |\phi_{n-1}(\theta)|,$$

that is

$$\max_{|x|\leqslant 1} |\phi_n(\theta)| \leqslant \max_{|x|\leqslant 1} |\phi_{n-1}(\theta)| \, [1 + 2Kc_n(n+1)].$$

Repeated application of this inequality shows that

$$\max_{|x|\leqslant 1} |\phi_n(\theta)| \leqslant \gamma_0 \exp\left\{2K \sum_{j=1}^{n} (j+1)c_j\right\}.$$

Hence by (4),

$$|\phi_n(\theta)| \leqslant \gamma_0(n+1)^{2KA} \quad (10)$$

for $-1 \leqslant x \leqslant 1$ and $n = 0, 1,\ldots$. Now we return to (7). Multiplying both sides by $e^{in\theta}$ and summing for $n = 0, 1,\ldots, m$, we obtain

$$e^{im\theta}\phi_m(\theta) = \sum_{n=0}^{m} \left\{\left[1 - 2\frac{\gamma_{n-1}}{\gamma_n}\right] p_n(x) - 2\alpha_{n-1} p_{n-1}(x) + \left[1 - 2\frac{\gamma_{n-2}}{\gamma_{n-1}}\right] p_{n-2}(x)\right\}.$$

Therefore, by (8) and (9),

$$|\phi_m(\theta)| \leqslant 2K(1 - x^2)^{-1/2} \sum_{n=0}^{m} c_n |\phi_n(\theta)|. \tag{11}$$

Using inequality (10), we get

$$|\phi_m(\theta)| \leqslant 2K\gamma_0(1 - x^2)^{-1/2} \sum_{n=0}^{m} c_n(n + 1)^{2KA}.$$

If $2KA < 1$, then by (4) and (9), the estimate (5) follows. Suppose that $2KA > 1$. Then using (4), we obtain

$$\begin{aligned} |\phi_m(\theta)| &\leqslant 2K\gamma_0(1 - x^2)^{-1/2} (m + 1)^{2KA-1} \sum_{n=0}^{m} c_n(n + 1) \\ &\leqslant 2KA\gamma_0(m + 1)^{2KA-1} \log(m + 1)(1 - x^2)^{-1/2} \end{aligned}$$

which is much better that (10). Now plug this inequality into (11). If $2KA - 1 < 1$, then (5) follows. Otherwise we get a new estimate which we again plug into (11). After finitely many similar steps we obtain

$$|\phi_m(\theta)| \leqslant B_1(1 - x^2)^{-B_2}$$

for $-1 \leqslant x \leqslant 1$ and $n = 1, 2,\ldots$ which, combined with (9), yields (5). The inequality (6) follows from (5) and Theorem 7.5 of [3].

Finally we note that the example of Jacobi polynomials shows that apart from the constants A_1 and A_2, our result cannot be improved.

References

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